Indian Statistical Institute
M.Math II Year
Second Semester Mid Semester Examination, 2004-2005
Fourier Analysis

Time: 3 hrs Date:15-03-05 Max. Marks : 70

Answer as many questions as you can. The maximum you can score is 70. Marks are indicated in the brackets. You may use your class-room notes in the exam. However you cannot use textbooks.

1. Find the Fourier series of the 2π-periodic function $f(x) = |\sin x|$. [10]

2. The Fourier coefficients of a 2π-periodic function f are given by:

$$\hat{f}(n) = \frac{1}{n \log|n|}, \quad |n| \geq 2.$$

Find a *reasonable* value of N such that

$$\int_{-\pi}^{\pi} |S_N{}^f(x) - f(x)|^2 \leq \frac{1}{27}.$$

(I don't want an absurd answer like $N = 10^5$!) Justify your answer. [10]

3. Let f be a function on $I\!R$ defined by:

$$\begin{aligned} f(x) &= 0 \text{ if } -\pi \leq x \leq 0 \\ &= x \text{ if } \quad 0 \leq x \leq \pi \\ &= 0 \text{ otherwise.} \end{aligned}$$

a) Without actually computing $\hat{f}(\lambda)$, explain why $\{t : \hat{f}(t) = 0\}$ has zero measure.
b) What is f_{per}? [8+7]

4. a) Find the function g on $I\!R$ such that

$$\hat{g}(\lambda) = \lambda^2 e^{-\lambda^2} + 3\lambda \, e^{-\lambda^2}.$$

b) If $f(x) = x \, e^{-(x-4)^2}$, what is $\hat{f}(\lambda)$? [8+7]

5. Let $g_n, g \in \mathcal{S}(I\!R)$. We say $g_n \xrightarrow{s} g$, if for every $k \geq 0, \ell \geq 0$, $(1+x^2)^\ell D^k g_n \to (1+x^2)^\ell D^k g$ uniformly on $I\!R$. Decide if the following is true or not:
$$g_n \xrightarrow{s} g \Rightarrow g_n \xrightarrow{L^2} g.$$
If true give a proof; if false give a counter example. [10]

6. Let $E = \bigcup_{n=2}^{\infty} \left[n, n + \frac{1}{n(\log n)^2}\right]$. Decide if the following statement is true or false:
There exists a *non-trivial* L^2-function g such that $\{x : g(x) \neq 0\}$ has finite measure and $\hat{g}$ is zero on E^c. [10]

7. You are given that both f and $\hat{f}$ are integrable on $I\!R$. Prove that f is in $L^2(I\!R)$. [10]

8. If $f \in \mathcal{S}$ and $g \in \mathcal{S}$ and $f * g \equiv 0$, does it necessarily mean that $f \equiv 0$ or $g \equiv 0$? Justify your answer. [10]